

POLETNI TABOR MATEMATIKA JE KUL How graphs can help us

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Vsebina

- 1 Drawing diagrams and Eulerian graphs
- 2 The Chess Knight problem and Hamiltonian graphs
- 3 Icosian game

Drawing diagrams

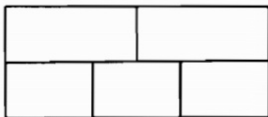
- Can we draw this diagram with just one move? The rule is that we have to draw each line exactly once.



- Is it important where we start to draw the diagram?

Drawing diagrams

- What about the next diagram? Can we draw it with just one move?



- How many moves do we need to draw it?

Drawing diagrams

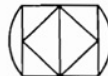
- How many moves do we need to draw each of these diagrams?



(a)



(b)



(c)



(d)

- How do we know at least how many moves do we need to draw an arbitrary diagram?

How graphs can help us?

Definicija

***Graf** sestavlja neprazna množica elementov, ki jih imenujemo **točke** ali **vozlišča** grafa, in seznam (neurejenih) parov teh elementov, ki jih imenujemo **povezave** grafa.*

Definition

A **graph** consists of a nonempty set of elements called **vertices** of a graph, and a list of (unordered) pairs of these elements called **edges** of a graph.

How graphs can help us?

Definicija

Dve ali več povezav, ki povezujejo isti par točk, imenujemo
vzporedne povezave oz. *večkratne povezave*.

Definition

Two or more edges connecting the same pair of vertices are called
multiple edges.

How graphs can help us?

Definicija

*Graf, ki je v enem kosu, je **povezan**, sicer pa je graf **nepovezan**. Povezavo povezanega grafa imenujemo **most**, če po njeni odstranitvi graf ni več povezan.*

Definition

A graph that is in one piece is said to be **connected** otherwise is said to be **disconnected**. If after deleting an edge in a connected graph the graph becomes disconnected then this edge is called a **bridge**.

How graphs can help us?

Definicija

Stopnja točke je število povezav, ki vsebujejo to točko.

Definition

The **degree** of a vertex is the number of edges incident to it, i.e. the number of edges that have it as an endpoint.

How graphs can help us?

Question: What can we say about the sum of the degrees of the vertices of a graph. Is this sum always an even number always an odd number or it depends on the graph?

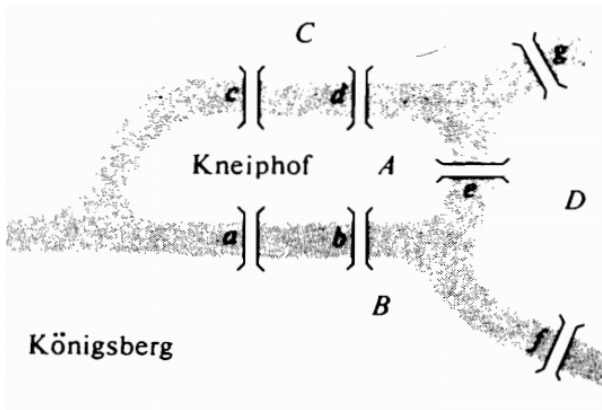
Answer: The sum of the degrees of the vertices of a graph is equal to twice the number of edges in a graph. So this sum is always an even number.

Question: What can we say about the number of vertices with odd degree in a graph? Even, odd, depends?

Answer: The number of vertices with odd degree in a graph is always an even number.

Eulerian (Euler) graphs

Four parts of the city named Königsberg (A, B, C, D) were connected with seven bridges (a, b, c, d, e, f, g).



Eulerian (Euler) graphs

Problem: Can we find a tour around the city of Königsberg so that we cross every bridge exactly once and we come back to the beginning of our walk?

Leonhard Euler (1707-1783) proved that this is impossible.

Eulerian (Euler) graphs

Definicija

*Povezan graf je **Eulerjev**, če se lahko sprehodimo po povezavah grafa na tak način, da prehodimo vsako povezavo natanko enkrat in se vrnemo na začetek sprehoda. Tak sprehod imenujemo **Eulerjev obhod**.*

Definition

A connected graph is **Eulerian** if we can make a walk in a graph such that we cross every edge exactly once and we come to the beginning of a walk. We call this walk an **Euler tour**.

Eulerian (Euler) graphs

Izrek

Povezan graf je Eulerjev natanko tedaj, ko ima vsaka točka v grafu sodo stopnjo.

Theorem

A connected graph is Eulerian if and only if every vertex in a graph has an even degree.

Question: At least how many additional bridges do we need to solve the problem of Königsberg city tour?

Eulerian (Euler) graphs

Question: Where is the connection between Eulerian graphs and drawing diagrams?

Answer: We take the diagram as it was a graph. If this graph is Eulerian then we can draw the diagram with one move.

Question: Is it true that if we can draw a diagram with one move then its graph must be Eulerian?

Answer: No.

Semi-Eulerian graphs

Definicija

*Povezan graf je **Poleulerjev**, če se lahko sprehodimo po povezavah grafa na tak način, da prehodimo vsako povezavo natanko enkrat. Tak sprehod imenujemo **Eulerjev sprehod**.*

Definition

A connected graph is **Semi-Eulerian** if we can make a walk in a graph such that we cross every edge exactly once. We call this walk an **Euler walk**.

Semi-Eulerian graphs

Izrek

Povezan graf je Poleulerjev natanko tedaj, ko ima natanko dve točki lihe stopnje.

Theorem

A connected graph is Semi-Eulerian if and only if exactly two vertices in a graph have an odd degree.

Drawing diagrams

Question: When can we draw the diagram with just one move?

Answer: We can draw the diagram with just one move if and only if its graph is Eulerian or Semi-Eulerian.

Drawing diagrams

Question: How to find an Eulerian tour in an Eulerian graph?

Answer: Fleury's algorithm.

Korak 1: Izberi začetno točko.

Step 1: Choose a starting vertex.

Korak 2: Prečkaj poljubno povezavo, le most izberi samo, kadar ni na voljo nobene druge povezave.

Step 2: Cross an arbitrary edge from this vertex, but choose a bridge only if there are no other edges to choose.

Korak 3: Prehojeno povezavo odstrani. Prav tako odstrani vse točke stopnje 0.

Step 3: Delete this edge and all vertices with degree 0.

Korak 4: Končaj, ko ni nobene povezave več.

Step 4: Finish when there are no more edges.

Drawing diagrams

Question: Can we use Fleury's algorithm to find Eulerian walk in a Semi-Eulerian graph?

Answer: Yes, but for starting point always take the vertex with odd degree.

Drawing diagrams

Question: How do we know at least how many moves do we need to draw an arbitrary diagram?

Answer: If we draw a graph for this diagram and if this graph has $2n$ vertices with odd degree, then we need at least n moves to draw this diagram.

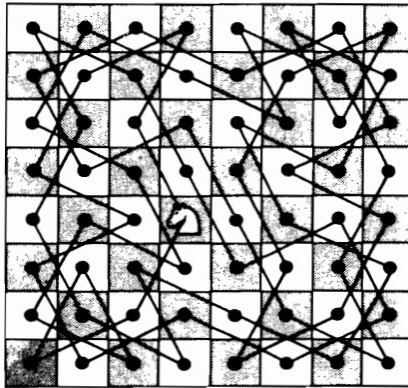
The chess knight problem

Problem: The chess knight is placed on the empty standard 8×8 chess board and, moving according to the rules of chess, must visit each square exactly once and must end on a square from which it began.

Question: Can we find a solution?

The chess knight problem

Answer: Yes.



The chess knight problem

Question: Can we find a solution for a 1×1 , 2×2 , 3×3 , 4×4 and 5×5 chess board?

Question: Can we find a solution for a reduced chess board 4×6 (squares with the sign X are forbidden for the knight)?

| | | | | | |
|---|---|---|---|---|---|
| X | X | X | | X | X |
| | | | | X | |
| X | X | | X | | X |
| | | | X | | X |

How graphs can help us?

Hamiltonian graphs

Definicija

*Sprehod v grafu pri katerem obiščemo vsako točko grafa natanko enkrat, nobene povezave ne prehodimo dvakrat in je končna točka enaka začetni točki imenujemo **hamiltonski cikel**. Če tak cikel v grafu obstaja, potem rečemo, da je graf **hamiltonski**.*

Definition

A walk in the graph such that every vertex is visited a single time, no edge is visited twice, and the ending point is the same as the starting point is called a **Hamiltonian cycle**. A connected graph is a **Hamiltonian graph** if there exists a Hamiltonian cycle.

The chess knight problem

Question: When does the solution to the chess knight problem exist according to Hamiltonian graphs?

Answer: The solution exists if and only if the graph corresponding to a chess board is Hamiltonian.

So we must check if a hamiltonian cycle in a graph exists or not. Sometimes this is an easy task, but usually is this a hard problem.

The chess knight problem

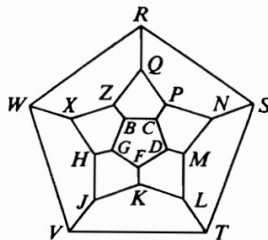
Question: Can we find a solution for a 1×1 , 2×2 , 3×3 , 4×4 and 5×5 chess board?

Exercise: Explain why we can't find a solution for a $n \times n$ chess board, where n is an odd integer not equal to 1?

Icosian game

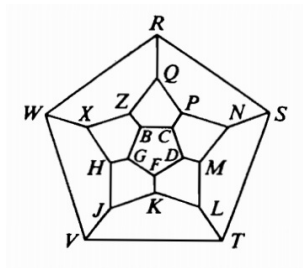
The game was invented by mathematician Sir William Rowan Hamilton (1805-1865).

Problem: How many Hamiltonian cycles with the first five vertices *BCPNM* can we find in a dodecahedron?



Icosian game

Problem: How many Hamiltonian cycles which starts with vertices *JVTSR* can we find in a dodecahedron?



Thank you.